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OPTIMAL DESIGN OF SDOF ENVELOPE SYSTEMS FOR THE RESPONSES OF MDOF DYNAMIC SYSTEMS

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SUMMARY

The concept of envelope system for a given dynamic system is proposed in this paper which refers to those systems whose module of transfer function in the whole range of frequency domain is always bigger than that of a given system. This concept opens a new way to study the problems of robust design and modelling for dynamic systems. The condition that an envelope system has to satisfy is rendered as the determination of the positiveness of a real polynomial function and Sturm's sequence method is used to establish an easily implemented criterion for evaluating the positiveness of the polynomial in terms of its coefficients. The optimization for the envelope system is expressed as the minimization of the 2-norm of its transfer function and simplex method is employed to search for the optimal solution. Two dynamic systems are used to illustrate the optimal design for the envelope systems of some of their responses. © 1998 John Wiley & Sons, Ltd.

KEY WORDS: envelope system; transfer function; robustness; Sturm's sequence; simplex method

INTRODUCTION

The dynamic performance is of considerable importance in the design of the structures which are vulnerable to dynamic loads, such as the precision machines, flexible space structures and aseismic structures. To enhance structural dynamic performance, it is required to do thorough structural dynamic analysis and choose better structural parameters and sometimes even required to mount an auxiliary control system on it particularly when it works in an uncertain environment like a space structure working in the space with various disturbances or an aseismic structure working under the threat of the unpredictable earthquake. Dynamic performance-based structural design considering the input uncertainty, structure uncertainty and even control uncertainty is an active research topic currently.[1,2]

Robustness is a measure of the stability of a dynamic system in the presence of system uncertainty and control uncertainty. A controller for a dynamic system is often designed to achieve the robustness requirement on the entire feedback control system.[3] In practice, it is not easy to realize the full state feedback especially when the dimension of the original dynamic system is very high. So an observer with reduced order needs to be designed, or in other words, a reduced model for the dynamic system needs to be established.

This paper, based on a novel concept of envelope system, presents a new way in robust modelling for a dynamic system. An envelope system of a given dynamic system refers to those systems whose module of transfer function in the whole range of frequency domain is always bigger than that of a given system. So any robustness requirement that the envelope system satisfies is surely satisfied by the original system. Obviously there are infinite number of envelope systems for a given dynamic system. But as its definition, the 2-norm of the envelope system's transfer function is lower-bounded by that of the original dynamic system. This means for a chosen structure of an envelope system a minimum 2-norm of its transfer function exists and can be

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achieved by optimally designing its parameters. Since there is no available state-space expression for the inequality of the modules of transfer functions involved in the definition of envelope system, the polynomial expression is used and that inequality turns out to be a positiveness requirement on a polynomial with real coefficients. To establish a criterion to evaluate the positiveness of a polynomial, Sturm's sequence method[4] is used. Consequently, the criterion is not a continuous function of the design variables but only an index. Its derivatives with respect to the design variables are meaningless. This shortcoming limits the choice of the optimization method used in optimal design for the envelope system parameters. In this paper, simplex method is chosen to carry out the optimization which just needs to evaluate the objective value and check the constrain conditions and does not need to calculate their derivatives. Finally, two dynamic systems are used to illustrate the optimal design for the envelope systems of some of their responses.

BASIC DEFINITIONS

Denote $H_o(\omega)$ as a transfer function of a given dynamic system $\mathbf{S}_o$ and assume it is stable and strictly proper. If a transfer function $H_e(\omega)$ satisfies the following condition:

$$|H_o(\omega)| < |H_e(\omega)| \tag{1}$$

or

$$H_o(\omega)H_o^{\dagger}(\omega) < H_e(\omega)H_e^{\dagger}(\omega) \tag{2}$$

it is called an envelope transfer function of $H_o(\omega)$ (where '†' means complex conjugate). If $H_e(\omega)$ is stable and strictly proper, it is called a stable and strictly proper envelope transfer function. Obviously, $H_e(\omega) = |H_o(\omega)|_{\infty}$ is an envelope transfer function of $H_o(\omega)$, but it is not a strictly proper one. An envelope system $\mathbf{S}_e$ of a given dynamic system $\mathbf{S}_o$ with respect to its transfer function $H_o(\omega)$ refers to those systems which have stable and strictly proper transfer functions that envelope $H_o(\omega)$.

If the polynomial expression is used for the transfer functions, the condition, equation (1) or (2), that an envelope transfer function has to satisfy will become an inequality about a polynomial function. This brings the following definition of the positiveness of a polynomial function. Denote $P(s)$ as a polynomial function of s with real coefficients. If it satisfies

$$P(s) > 0 \quad (s \in R = (-\infty,\ \infty)), \tag{3}$$

it is called a positive polynomial function. Obviously, when $P(s)$ is an even function, it is positive only if $P(s) > 0 (s \in R^+ = [0,\ \infty))$. From equation (2) it can be seen that the polynomial of interest for the envelope system is always an even function with real coefficients. In the following section an easily implemented criterion to evaluate the positiveness of a polynomial on R^+ will be established.

POSITIVENESS CRITERION FOR POLYNOMIALS WITH REAL COEFFICIENTS

Sturm's sequence method is a well-known method for calculating the number of real roots of a real polynomial in a specific interval. It states[4]

For a real polynomial $P(s)$, its Sturm's sequence is obtained by the following procedure:

$$\begin{aligned} P(s) &= P_1(s)Q_1(s) - P_2(s) \\ P_1(s) &= P_2(s)Q_2(s) - P_3(s) \\ P_2(s) &= P_3(s)Q_3(s) - P_4(s) \\ &\vdots \\ P_{n-2}(s) &= P_{n-1}(s)Q_{n-1}(s) - P_n \end{aligned} \tag{4}$$

where $P_1(s) = P'(s)$ (the first derivative of $P(s)$), P_n is a constant and the order of $P_i(s)$ is always lower than that of $P_{i-1}(s)$. Denote the count of sign change of the sequence $[P(s), P_1(s), P_2(s), \ldots, P_{n-1}(s), P_n]$ as $N_P(s)$. Then the number of real roots of $P(s)$ in interval $[a, b]$ is

$$N_r = N_P(a) - N_P(b) \tag{5}$$

Denote $a_{1,0}$ as the constant term in $P(s)$ so do $a_{1,i}$ $(i = 1, 2, \ldots, n)$ in $P_i(s)$ $(i = 1, 2, \ldots, n)$. Similarly, denote $a_{n+1,0}$ as the coefficient in the highest-order term in $P(s)$ so do $a_{n-i+1,i}$ $(i = 1, 2, \ldots, n)$ in $P_i(s)$ $(i = 1, 2, \ldots, n)$. Therefore, $N_P(0)$ and $N_P(\infty)$ are equal to the counts of sign change of the sequences $[a_{1,0}, a_{1,1}, a_{1,2}, \ldots, a_{1,n}]$ and $[a_{n+1,0}, a_{n,1}, a_{n-1,2}, \ldots, a_{1,n}]$, respectively. The following will derive a simple formulation to calculate the above two sequences.

Let $a_{j,i}$ $(j = 1, 2, \ldots, n - i + 1)$ be the coefficients of $P_i(s)$, i.e.

$$P_i(s) = \sum_{j=1}^{n-i+1} a_{j,i} s^{i-1} \tag{6}$$

The procedure to calculate $a_{j,i+1}$ $(i = 1, 2, \ldots, n-1)$ from $a_{j,i}$ and $a_{j,i-1}$ as specified by equation (4) can be expressed as the Gauss elimination for the last two elements in the last row of the following matrix:

$$\begin{bmatrix} a_{1,i} & a_{2,i} & \cdots & a_{n-i,i} & a_{n-i+1,i} & \\ & a_{1,i} & a_{2,i} & \cdots & a_{n-i,i} & a_{n-i+1,i} \\ a_{1,i-1} & a_{2,i-1} & \cdots & a_{n-i,i} & a_{n-i+1,i-1} & a_{n-i+2,i-1} \end{bmatrix} \tag{7}$$

Then $a_{j,i+1}$ $(i = 1, 2, \ldots, n-1)$ is the negative of the last row in the obtained matrix after the Gauss elimination.

The explicit expression of $a_{j,i+1}$ $(i = 1, 2, \ldots, n-1)$ is

$$a_{j,i+1} = -(b_j^{(1)} - b_j^{(0)} b_{n-i+1}^{(1)}) \quad (j = 1, 2, \ldots, n-i) \tag{8}$$

where

$$b_j^{(1)} = a_{j,i-1} - b_{j-1}^{(0)} a_{n-i+2,i-1} \quad (j = 1, 2, \ldots, n-i+1) \tag{9}$$

$$b_j^{(0)} = a_{j,i}/a_{n-i+1,i} \quad (j = 1, 2, \ldots, n-i) \tag{10}$$

and $b_0^{(0)} = 0$. In the above equations, $a_{j,0}$ $(j = 1, 2, \ldots, n+1)$ are known from the definition of $P(s)$ and because of the relationship $P_1(s) = P'(s)$, $a_{j,1}$ $(j = 1, 2, \ldots, n)$ are

$$a_{j,1} = j a_{j+1,0} \quad (j = 1, 2, \ldots, n) \tag{11}$$

Once the calculation procedure described by equation (8)–(10) is finished, $N_P(0)$ and $N_P(\infty)$ can then be counted. So the number of roots of $P(s)$ on R^+ is

$$N_{r+} = N_P(0) - N_P(\infty) \tag{12}$$

Finally, the criterion to evaluate the positiveness of a polynomial is

$$a_{1,0} > 0 \quad \text{and} \quad N_{r+} = 0. \tag{13}$$

The first condition in the above equation means that $P(s)$ passes through a positive point and the second means it does not intersect with the right half real axis. Therefore, it must be positive on R^+ because of its continuity.

OPTIMAL ENVELOPE SYSTEM DESIGN

Suppose the transfer function of a given dynamic system can be expressed in a rational polynomial form as follows:

$$H_o(\omega) = \frac{\sum_{k=1}^{n_{Ro,p}+1} p_{Ro,k}\omega^{2(k-1)} + i\omega \sum_{k=1}^{n_{Io,p}+1} p_{Io,k}\omega^{2(k-1)}}{\sum_{k=1}^{n_{Ro,q}+1} q_{Ro,k}\omega^{2(k-1)} + i\omega \sum_{k=1}^{n_{Io,q}+1} q_{Io,k}\omega^{2(k-1)}}. \tag{14}$$

The square module of $H_o(\omega)$ is

$$H_o(\omega)H_o^\dagger(\omega) = \frac{(\sum_{k=1}^{n_{Ro,p}+1} p_{Ro,k}\omega^{2(k-1)})^2 + (\omega \sum_{k=1}^{n_{Io,p}+1} p_{Io,k}\omega^{2(k-1)})^2}{(\sum_{k=1}^{n_{Ro,q}+1} q_{Ro,k}\omega^{2(k-1)})^2 + (\omega \sum_{k=1}^{n_{Io,q}+1} q_{Io,k}\omega^{2(k-1)})^2}. \tag{15}$$

Rearranging the terms in the numerator and denominator of equation (15), it can also be expressed as

$$H_o(\omega)H_o^\dagger(\omega) = \frac{\sum_{k=1}^{n_{o,p}+1} p_{o,k}\omega^{2(k-1)}}{\sum_{k=1}^{n_{o,q}+1} q_{o,k}\omega^{2(k-1)}}, \tag{16}$$

where the definitions of $n_{o,p}$, $n_{o,q}$, $p_{o,k}$ and $q_{o,k}$ are given in Appendix I.

Similarly, if the transfer function of an envelope system is also expressed in a rational polynomial form as follows:

$$H_e(\omega) = \frac{\sum_{k=1}^{n_{Re,p}+1} p_{Re,k}\omega^{2(k-1)} + i\omega \sum_{k=1}^{n_{Ie,p}+1} p_{Ie,k}\omega^{2(k-1)}}{\sum_{k=1}^{n_{Re,q}+1} q_{Re,k}\omega^{2(k-1)} + i\omega \sum_{k=1}^{n_{Ie,q}+1} q_{Ie,k}\omega^{2(k-1)}}, \tag{17}$$

according to equation (15), its square module can be written as

$$H_e(\omega)H_e^\dagger(\omega) = \frac{(\sum_{k=1}^{n_{Re,p}+1} p_{Re,k}\omega^{2(k-1)})^2 + (\omega \sum_{k=1}^{n_{Ie,p}+1} p_{Ie,k}\omega^{2(k-1)})^2}{(\sum_{k=1}^{n_{Re,q}+1} q_{Re,k}\omega^{2(k-1)})^2 + (\omega \sum_{k=1}^{n_{Ie,q}+1} q_{Ie,k}\omega^{2(k-1)})^2}. \tag{18}$$

Following the same procedure in obtaining equation (16), the simple rational polynomial form for equation (18) is

$$H_e(\omega)H_e^\dagger(\omega) = \frac{\sum_{k=1}^{n_{e,p}+1} p_{e,k}\omega^{2(k-1)}}{\sum_{k=1}^{n_{e,q}+1} q_{e,k}\omega^{2(k-1)}} \tag{19}$$

where $n_{e,p}$, $n_{e,q}$, $p_{e,k}$ and $q_{e,k}$ can be calculated from the coefficients in equation (18) using the same equations given in Appendix I.

Substituting equations (16) and (19) in equation (2), the polynomial expression for the inequality is

$$\frac{\sum_{k=1}^{n_{e,p}+1} p_{e,k}\omega^{2(k-1)}}{\sum_{k=1}^{n_{e,q}+1} q_{e,k}\omega^{2(k-1)}} > \frac{\sum_{k=1}^{n_{o,p}+1} p_{o,k}\omega^{2(k-1)}}{\sum_{k=1}^{n_{o,q}+1} q_{o,k}\omega^{2(k-1)}} \tag{20}$$

or

$$\frac{\sum_{k=1}^{n_{e,p}+1} p_{e,k}\omega^{2(k-1)} \sum_{k=1}^{n_{o,p}+1} q_{o,k}\omega^{2(k-1)} - \sum_{k=1}^{n_{e,p}+1} q_{e,k}\omega^{2(k-1)} \sum_{k=1}^{n_{o,p}+1} p_{o,k}\omega^{2(k-1)}}{\sum_{k=1}^{n_{e,q}+1} q_{e,k}\omega^{2(k-1)} \sum_{k=1}^{n_{o,p}+1} q_{o,k}\omega^{2(k-1)}} > 0. \tag{21}$$

Because the denominator in equation (21) is always greater than zero, it is then required that its numerator must be greater than zero, i.e.

$$\sum_{k=1}^{n_{e,p}+1} p_{e,k}\omega^{2(k-1)} \sum_{k=1}^{n_{o,q}+1} q_{o,k}\omega^{2(k-1)} - \sum_{k=1}^{n_{e,q}+1} q_{e,k}\omega^{2(k-1)} \sum_{k=1}^{n_{o,p}+1} p_{o,k}\,\omega^{2(k-1)} > 0 \tag{22}$$

or

$$P(\omega) = \sum_{k=1}^{n+1} a_k\omega^{2(k-1)} > 0 \tag{23}$$

where n and a_k are given in Appendix II.

The objective function chosen to optimize the envelope system is the 2-norm of its transfer function which is defined as

$$\mathbf{E} = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\mathrm{e}}(\omega) H_{\mathrm{e}}^{\dagger}(\omega)\,\mathrm{d}\omega. \tag{24}$$

Finally, combining equations (13), (17), (23) and (24), the optimal envelope system design problem can be expressed as

Find

$$p_{Re,k}(k = 1, 2, \ldots, n_{Re,p}), q_{Re,k}(k = 1, 2, \ldots, n_{Re,q})$$

$$p_{Ie,k}(k = 1, 2, \ldots, n_{Ie,p}), q_{Ie,k}(k = 1, 2, \ldots, n_{Ie,q})$$

to minimize **E**

subject to $$a_1 > 0 \text{ and } N_{r+} = 0 \tag{25}$$

where N_{r+} is the number of real roots on R^+ of the polynomial equation (23).

OPTIMAL SDOF ENVELOPE SYSTEM DESIGN FOR THE RESPONSES OF MDOF DYNAMIC SYSTEMS

The dynamic equation of a MDOF system can be generally expressed as

$$mx''(t) + cx'(t) + kx(t) = QW(t) \tag{26}$$

where m and k are system's mass and stiffness matrices, and c its damping matrix which is assumed of Rayleigh's type here. $W(t)$ is a single input to the system. $x''(t)$, $x'(t)$ and $x(t)$ are system's acceleration, velocity and displacement vectors. The response of interest is generally expressed as

$$z(t) = d_d^{\mathrm{T}} x(t) + d_v^{\mathrm{T}} x'(t). \tag{27}$$

Denote v_m $(m = 1, 2, \ldots, n)$ as the eigenvectors of the system and $y_m(t)$ $(m = 1, 2, \ldots, n)$ as their corresponding generalized co-ordinates. By making the mode decomposition for equation (26), the dynamic equations governing the generalized co-ordinates can be expressed as

$$y_m''(t) + 2\xi_m \omega_m y_m'(t) + \omega_m^2 y_m(t) = v_m^{\mathrm{T}} Q W(t) \tag{28}$$

Using Fourier transformation, the frequency component of $y_m(t)$ can be obtained as

$$y_m(\omega) = \frac{1}{G_m(\omega)} v_m^{\mathrm{T}} Q W(\omega) \tag{29}$$

where

$$G_m(\omega) = \omega_m^2 - \omega^2 + 2\xi_m \omega_m \omega \mathrm{i} \tag{30}$$

With equations (27) and (29), the frequency component of the response can be derived as

$$z(\omega) = \left(\sum_{m=1}^{n} \frac{\lambda_{dm}}{G_m(\omega)} + \mathrm{i}\omega \sum_{m=1}^{n} \frac{\lambda_{vm}}{G_m(\omega)} \right) W(\omega), \tag{31}$$

where

$$\lambda_{dm} = (v_m^{\mathrm{T}} d_d)(v_m^{\mathrm{T}} Q), \tag{32}$$

$$\lambda_{vm} = (v_m^{\mathrm{T}} d_v)(v_m^{\mathrm{T}} Q). \tag{33}$$

Therefore, the transfer function from the input $W(t)$ to the output $z(t)$ is

$$H_{Wz}(\omega) = \frac{\sum_{m=1}^{n} (\lambda_{dm} + \mathrm{i}\omega\lambda_{vm}) \prod_{j=1(j\neq m)}^{n} G_j(\omega)}{\prod_{j=1}^{n} G_j(\omega)} \tag{34}$$

Carry out the polynomial multiplications in the numerator and denominator in the above equation and express them in the form as follows:

$$\prod_{j=1}^{n} G_j(\omega) = \sum_{j=1}^{n+1} g_{R,j}\omega^{2(j-1)} + \mathrm{i}\omega \sum_{j=1}^{n} g_{I,j}\omega^{2(j-1)} \tag{35}$$

$$\prod_{j=1(j\neq m)}^{n} G_j(\omega) = \sum_{j=1}^{n} g_{Rm,j}\omega^{2(j-1)} + \mathrm{i}\omega \sum_{j=1}^{n-1} g_{Im,j}\omega^{2(j-1)} \tag{36}$$

where the coefficients $g_{R,j}$, $g_{I,j}$, $g_{Rm,j}$ and $g_{Im,j}$ can be calculated using the recursive formulas given in Appendix III.

Finally, the rational polynomial expression for $H_{Wz}(\omega)$ which has the same form as equation (14) is

$$H_{Wz}(\omega) = \frac{\sum_{j=1}^{n} g_{R\mathrm{o},j}\omega^{2(j-1)} + \mathrm{i}\omega \sum_{j=1}^{n} g_{I\mathrm{o},j}\omega^{2(j-1)}}{\sum_{j=1}^{n+1} g_{R,j}\omega^{2(j-1)} + \mathrm{i}\omega \sum_{j=1}^{n} g_{I,j}\omega^{2(j-1)}} \tag{37}$$

where $g_{R\mathrm{o},j}$ and $g_{I\mathrm{o},j}$ are given in Appendix IV.

The dynamic equation of a SDOF envelope system is expressed as

$$x_{\mathrm{e}}''(t) + 2\xi_{\mathrm{e}}\omega_{\mathrm{e}}x_{\mathrm{e}}'(t) + \omega_{\mathrm{e}}^2 x_{\mathrm{e}}(t) = W(t) \tag{38}$$

where ξ_{e}, ω_{e} are design variables. $x_{\mathrm{e}}''(t)$, $x_{\mathrm{e}}'(t)$ and $x_{\mathrm{e}}(t)$ are the acceleration, velocity and displacement states of the envelope system. Its response is expressed as

$$z_{\mathrm{e}}(t) = A_d x_{\mathrm{e}}(t) + A_v x_{\mathrm{e}}'(t) \tag{39}$$

where A_d and A_v are also design variables. It is not difficult to derive the transfer function from the input to the output for the envelope system. Its expression which is the same as equation (17) is as follows:

$$H_{Wz_{\mathrm{e}}}(\omega) = \frac{A_d + \mathrm{i}\omega A_v}{\omega_{\mathrm{e}}^2 - \omega^2 + \mathrm{i}\omega(2\xi_{\mathrm{e}}\omega_{\mathrm{e}})} \tag{40}$$

The explicit expression of the 2-norm of the envelope transfer function is

$$\mathbf{E} = \frac{A_d^2}{4\xi_{\mathrm{e}}\omega_{\mathrm{e}}^3} + \frac{A_v^2}{4\xi_{\mathrm{e}}\omega_{\mathrm{e}}} \tag{41}$$

Combining equation (25) and equations (37), (40) and (41), the optimal SDOF envelope system design for the responses of MDOF dynamic systems can then be described as

$$\begin{aligned} &\text{Find} && \xi_{\mathrm{e}}, \omega_{\mathrm{e}}, A_d \text{ and } A_v \\ &\text{to minimize} && (A_d^2/\omega_{\mathrm{e}}^2 + A_v^2)/4\xi_{\mathrm{e}}\omega_{\mathrm{e}} \\ &\text{subject to} && a_1 > 0 \text{ and } N_{\mathrm{r}+} = 0 \end{aligned} \tag{42}$$

where a_1 and $N_{\mathrm{r}+}$ are calculated with respect to the transfer functions (37) and (40).

SOLUTION PROCEDURE AND APPLICATIONS

As the constrain function N_{r+} in the optimization problem (25) or (42) is only a discrete index. Its derivatives with respect to the design variables are meaningless. This shortcoming limits the choice of the optimization method for finding the optimal solution. In this paper, simplex method is used which just needs to calculate the objective value and check the satisfaction of the constrain conditions but does not need to calculate the derivatives of both the objective and constrain functions.

The solution procedure using simplex method is as follows:

Step 1. Find a feasible solution $\mathbf{D}_0$;

Step 2. Do one-dimensional search in every conjugate direction starting from $\mathbf{D}_0$ and denote the obtained feasible solutions as $\mathbf{D}_i$ ($i = 1, 2, \ldots, n_d$, where n_d is the number of the design variables);

Step 3. Use the $n_d + 1$ feasible solutions to form a simplex in the feasible domain and then employ the standard procedure to renew the simplex by replacing the vertex having the largest objective value with a new one obtained by doing one-dimensional search in the direction connecting that vertex and the centre point of the rest;

Step 4. Repeat the above renewal process until the solution is convergent.

Example 1. Design SDOF envelope systems for the responses of a two degrees of freedom system described by the following dynamic equations:

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} + \begin{bmatrix} 0{\cdot}201 & -0{\cdot}2 \\ -0{\cdot}2 & 0{\cdot}401 \end{bmatrix}\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} + \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} W(t) \tag{43}$$

First, the response of interest is chosen as $z(t) = x_1(t)$. After the optimization search, the optimal parameters for its SDOF envelope system are found to be $\xi_e = 0{\cdot}02541$, $\omega_e = 6{\cdot}186$ and $A_d = 4{\cdot}759$. The square modules of both transfer functions are shown in Figure 1 from which the clear enveloping nature of

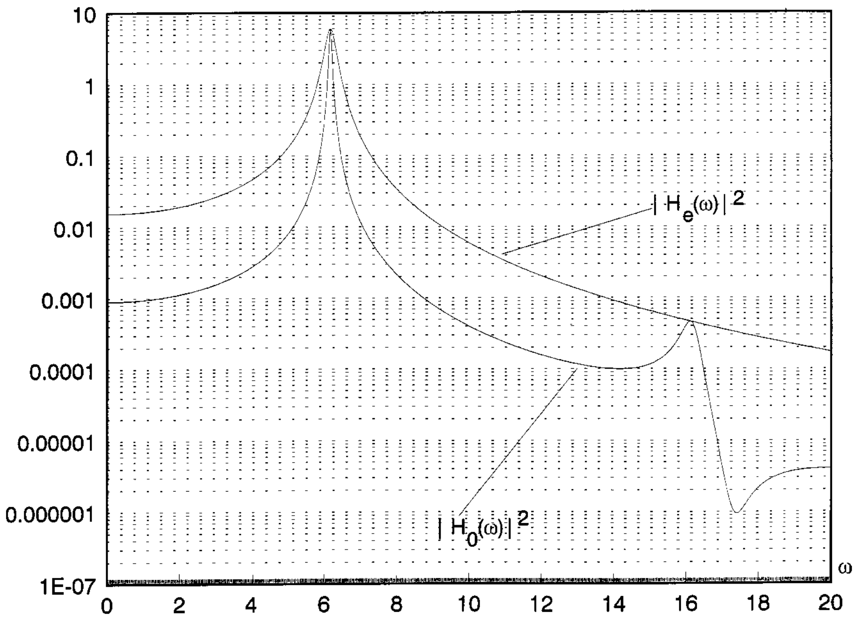

Figure 1. Square modules of the original transfer function and its envelope transfer function

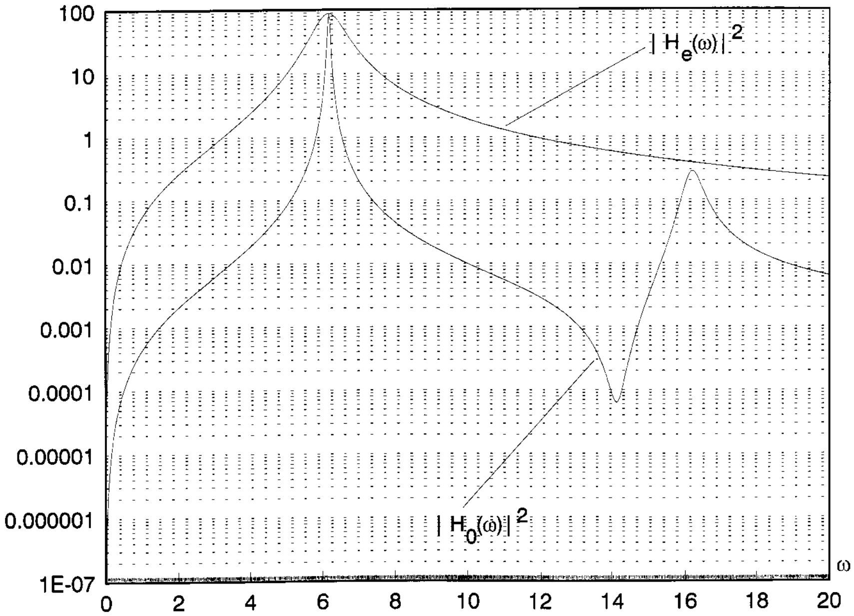

Figure 2. Square modules of the original transfer function and its envelope transfer function

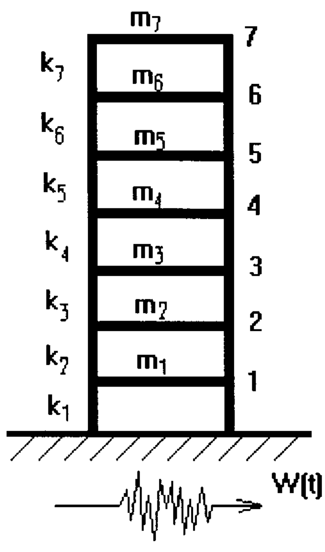

Figure 3. A seven-storey plane shear frame building

$H_e(\omega)$ can be perceived. The 2-norm of $H_o(\omega)$ is calculated using the available method[5,6] which is 0·2320 whereas that of $H_e(\omega)$ is 0·9412.

Next, the response is chosen to be $z(t) = x_2'(t)$. In this case, the optimal parameters for its SDOF envelope system converge to $\xi_e = 0{\cdot}07452$, $\omega_e = 6{\cdot}188$ and $A_v = 8{\cdot}624$. The Square modules of both transfer functions are shown in Figure 2. The 2-norm values of $H_o(\omega)$ and $H_e(\omega)$ are 3·456 and 40·33, respectively.

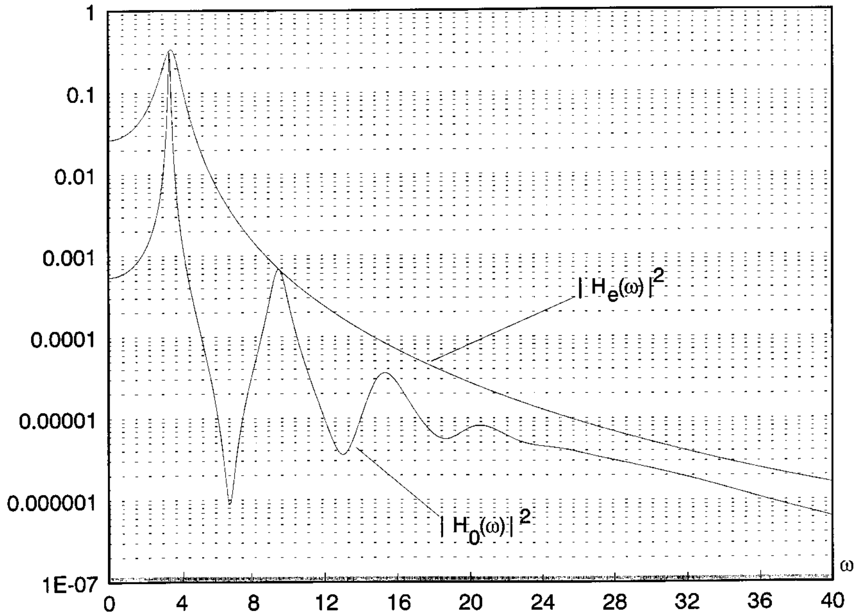

Figure 4. Square modules of the original transfer function and its envelope transfer function ($\alpha = 0{\cdot}0$; $\beta = 0{\cdot}01$)

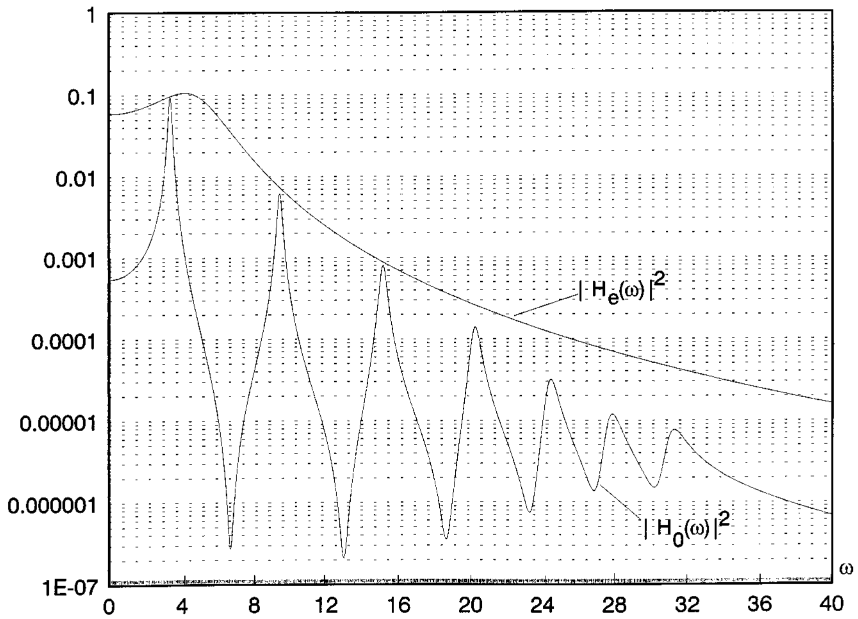

Figure 5. Square modules of the original transfer function and its envelope transfer function ($\alpha = 0{\cdot}2$; $\beta = 0{\cdot}001$)

Example 2. A seven-storey plane shear frame building is shown in Figure 3. The structural parameters are: $m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m_7 = 1{\cdot}0$ and $k_1 = 300{\cdot}0$, $k_2 = 280{\cdot}0$, $k_3 = 260{\cdot}0$, $k_4 = 240{\cdot}0$, $k_5 = 220{\cdot}0$, $k_6 = 200{\cdot}0$, $k_7 = 180{\cdot}0$.

Table I. Results for Example 2

Rayleigh's damping		Envelope system parameters			2-norm values	
α	β	ξ_e	ω_e	A_d	$H_e(\omega)$	$H_o(\omega)$
0·0	0·01	0·1416	3·515	2·013	0·1648	0·01927
0·2	0·001	0·4078	5·109	6·327	0·1840	0·01133

The output concerned is the displacement response of the top floor. Two sets of Rayleigh's damping parameters are chosen so as to make different shapes of $H_o(\omega)$. The optimal parameters of the SDOF envelope system as well as the 2-norm values of both transfer functions are calculated and recorded in Table I. The square modules of the transfer functions in both cases are shown in Figures 4 and 5.

It can be seen from Figure 4 that the designed SDOF envelope system is controlled by the contributions from the first and second vibration modes in the concerned response. However in Figure 5, the contributions from the first and third vibration modes dominate the characters of the SDOF envelope system. That explains why the 2-norm value of the SDOF envelope transfer function in the second case is greater than that in the first one even the 2-norm values of their original transfer functions are the other way round.

CONCLUSIONS

A novel concept of envelope system is introduced in this paper. The optimal design for such a system is formulated and applied to two dynamic systems. Although only the explicit formulations are given for the optimal design of SDOF envelope systems, it is not difficult to obtain those for MDOF envelope systems. This concept is especially useful in the robust modelling for high-dimensional dynamic systems which often have densely distributed eigenvalues. Also, it has potential application on robust control design for dynamic systems using robust reduced models.

APPENDIX I

Definitions of $n_{o,p}$, $n_{o,q}$, $p_{o,k}$ and $q_{o,k}$ in equation (16)

$$n_{o,p} = \begin{cases} 2n_{Io,p} + 1, & n_{Ro,p} \leqslant n_{Io,p} \\ 2n_{Ro,p}, & n_{Ro,p} > n_{Io,p} \end{cases} \tag{44}$$

If $n_{Ro,p} \leqslant n_{Io,p}$,

$$p_{o,k} = \begin{cases} d_{Rp,1}, & k = 1 \\ d_{Rp,k} + d_{Ip,k}, & 1 < k \leqslant 2n_{Ro,p} + 1 \\ d_{Ip,k}, & k > 2n_{Ro,p} + 1 \end{cases} \tag{45}$$

else if $n_{Ro,p} > n_{Io,p}$,

$$p_{o,k} = \begin{cases} d_{Rp,1}, & k = 1 \\ d_{Rp,k} + d_{Ip,k}, & 1 < k \leqslant 2n_{Io,p} + 2 \\ d_{Ip,k}, & k > 2n_{Io,p} + 2, \end{cases} \tag{46}$$

where

$$d_{Rp,k} = \begin{cases} \sum_{j=1}^{k} p_{R\mathrm{o},j} p_{R\mathrm{o},k-j+1}, & k = 1, 2, \ldots, n_{R\mathrm{o},p} + 1 \\ \sum_{j=k-n_{R\mathrm{o},p}}^{n_{R\mathrm{o},p}+1} p_{R\mathrm{o},j} p_{R\mathrm{o},k-j+1}, & k = n_{R\mathrm{o},p} + 2, n_{R\mathrm{o},p} + 3, \ldots, 2n_{R\mathrm{o},p} + 1, \end{cases} \tag{47}$$

$$d_{Ip,k} = \begin{cases} \sum_{j=1}^{k} p_{I\mathrm{o},j} p_{I\mathrm{o},k-j+1}, & k = 1, 2, \ldots, n_{I\mathrm{o},p} + 1 \\ \sum_{j=k-n_{I\mathrm{o},p}}^{n_{I\mathrm{o},p}+1} p_{I\mathrm{o},j} p_{I\mathrm{o},k-j+1}, & k = n_{I\mathrm{o},p} + 2, n_{I\mathrm{o},p} + 3, \ldots, 2n_{I\mathrm{o},p} + 1. \end{cases} \tag{48}$$

The above formulations can also be used to define $n_{\mathrm{o},q}$ and $q_{\mathrm{o},k}$.

APPENDIX II

Definitions of n and a_k in equation (23)

Suppose $n_{\mathrm{e},p} \leqslant n_{\mathrm{o},q}$. Then the coefficients in the polynomial function that corresponds to the first term in equation (22) are

$$b_{pq,k} = \begin{cases} \sum_{j=1}^{k} p_{\mathrm{e},j} q_{\mathrm{o},k-j+1}, & k \leqslant n_{\mathrm{e},p} + 1 \\ \sum_{j=1}^{n_{\mathrm{e},p}+1} p_{\mathrm{e},j} q_{\mathrm{o},k-j+1}, & n_{\mathrm{e},p} + 1 < k \leqslant n_{\mathrm{o},q} + 1 \\ \sum_{j=k-n_{\mathrm{o},p}}^{n_{\mathrm{e},p}+1} p_{\mathrm{e},j} q_{\mathrm{o},k-j+1}, & n_{\mathrm{o},q} + 1 < k \leqslant n_{\mathrm{e},p} + n_{\mathrm{o},q} + 1 \end{cases} \tag{49}$$

Similarly if $n_{\mathrm{e},q} \leqslant n_{\mathrm{o},p}$, the coefficients in the polynomial function that corresponds to the second term in equation (22) are

$$b_{qp,k} = \begin{cases} \sum_{j=1}^{k} q_{\mathrm{e},j} p_{\mathrm{o},k-j+1}, & k \leqslant n_{\mathrm{e},q} + 1 \\ \sum_{j=1}^{n_{\mathrm{e},q}+1} q_{\mathrm{e},j} p_{\mathrm{o},k-j+1}, & n_{\mathrm{e},q} + 1 < k \leqslant n_{\mathrm{o},p} + 1 \\ \sum_{j=k-n_{\mathrm{o},p}}^{n_{\mathrm{e},q}+1} q_{\mathrm{e},j} p_{\mathrm{o},k-j+1}, & n_{\mathrm{o},p} + 1 < k \leqslant n_{\mathrm{e},q} + n_{\mathrm{o},q} + 1 \end{cases} \tag{50}$$

Let $n = \max(n_{\mathrm{e},p} + n_{\mathrm{o},q}, n_{\mathrm{e},q} + n_{\mathrm{o},p})$. The definitions of $a_k (k = 1, 2, \ldots, n + 1)$ are

$$a_k = \begin{cases} b_{pq,k} - b_{qp,k}, & k \leqslant \min(n_{\mathrm{e},p} + n_{\mathrm{o},q} + 1, n_{\mathrm{e},q} + n_{\mathrm{o},p} + 1) \\ b_{pq,k}, & k > \min(n_{\mathrm{e},p} + n_{\mathrm{o},p} + 1, n_{\mathrm{e},p} + n_{\mathrm{o},p} + 1);\ n_{\mathrm{e},p} + n_{\mathrm{o},q} > n_{\mathrm{e},q} + n_{\mathrm{o},p} \\ -b_{qp,k}, & k > \min(n_{\mathrm{e},p} + n_{\mathrm{o},p} + 1, n_{\mathrm{e},q} + n_{\mathrm{o},p} + 1);\ n_{\mathrm{e},p} + n_{\mathrm{o},q} < n_{\mathrm{e},q} + n_{\mathrm{o},p} \end{cases} \tag{51}$$

APPENDIX III

Recursive calculation for $g_{R,j}$ and $g_{I,j}$ in equation (35)

Use the following notation,

$$G^{(m)}(\omega) = \prod_{j=1}^{m} G_j(\omega) = \sum_{k=1}^{m+1} g_{R,k}^{(m)} \omega^{2(k-1)} + \mathrm{i}\omega \sum_{k=1}^{m} g_{I,k}^{(m)} \omega^{2(k-1)}. \tag{52}$$

Then the recursive formula to calculate $g_{R,k}^{(m+1)}$ and $g_{I,k}^{(m+1)}$ from $g_{R,k}^{(m)}$, $g_{I,k}^{(m)}$, ξ_{m+1} and ω_{m+1} are,

$$g_{R,k}^{(m+1)} = \begin{cases} -g_{R,m+1}^{(m)} & k = m + 2 \\ -g_{R,k-1}^{(m)} + \omega_{m+1}^2 g_{R,k}^{(m)} - 2\xi_{m+1}\omega_{m+1} g_{I,k-1}^{(m)} & 1 < k < m + 2 \\ \omega_{m+1}^2 g_{R,1}^{(m)} & k = 1 \end{cases} \tag{53}$$

and

$$g_{I,k}^{(m+1)} = \begin{cases} -g_{I,m}^{(m)} + 2\xi_{m+1}\omega_{m+1}g_{R,m+1}^{(m)}, & k = m+1 \\ -g_{I,k-1}^{(m)} + \omega_{m+1}^2 g_{I,k}^{(m)} + 2\xi_{m+1}\omega_{m+1}g_{R,k}^{(m)}, & 1 < k < m+1 \\ \omega_{m+1}^2 g_{I,1}^{(m)} + 2\xi_{m+1}\omega_{m+1}g_{R,1}^{(m)}, & k = 1 \end{cases} \tag{54}$$

APPENDIX IV

Definition of $g_{Ro,j}$ and $g_{Io,j}$ in equation (37)

$$g_{Ro,j} = \begin{cases} \sum_{m=1}^{n} (\lambda_{dm} g_{Rm,j} - \lambda_{vm} g_{Im,j-1}), & 1 < j \leqslant n \\ \sum_{m=1}^{n} \lambda_{dm} g_{Rm,1}, & j = 1, \end{cases} \tag{55}$$

$$g_{Io,j} = \begin{cases} \sum_{m=1}^{n} \lambda_{vm} g_{Rm,n}, & j = n \\ \sum_{m=1}^{n} (\lambda_{dm} g_{Im,j} + \lambda_{vm} g_{Rm,j}), & 1 \leqslant j < n \end{cases} \tag{56}$$

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